Malaysian Journal of Mathematical Sciences 16(1): 79–86 (2022)
https://doi.org/10.47836/mjms.16.1.07

Malaysian Journal of Mathematical Sciences

Journal homepage: https://einspem.upm.edu.my/journal

Solving Fredholm Second Order Integro-Differential Equation with Logarithmic Kernel Using the Airfoil Collocation Method

Ramdani, N. E. *[1] and Hadj, A.[2]

[1] *Department of Civil Engineering, Faculty of Technology, University of Saad Dahlab Blida, Algeria*
[2] *Applied Mathematics Laboratory, University Badji Mokhtar-Annaba, Algeria*

E-mail: nedjemeddine.ramdani@yahoo.com
**Corresponding author*

Received: 11 June 2021
Accepted: 1 October 2021

Abstract

In this paper, the Airfoil polynomials for solving the second order integro-differential equation with a singular kernel is considered. The collocation method is developed to obtain an approximate solution of the equation. We present an error analysis and conclude by providing numerical tests to verify our results.

Keywords: Fredholm integro-differential equation; second order; logarithmic kernel; airfoil polynomials; collocation method.

1 Introduction

The appearance of integral equations can be traced back at least to the discovery of the integral theorem by Fourier. Indeed, the recasting of Ivar Fredholm's theory of linear integral equations of the second kind by David Hilbert and Erhard Schmidt in the first decade of the last century formed an important foundation for the development of modern mathematics.

Integral equations method appears in many models describing various phenomena in several fields of applied science and engineering such as: economics, fluid dynamics, electrodynamics, elasticity, fracture mechanics, biology, scattering of surface water waves, other scientific fields and the latest high technology. Several techniques have been developed for solving such equations including: Legendre-Galerkin method [12], shifted Legendre projection method [5], Piecewise linear approximation and Polynomial approximation [8], B-spline method [9], Meshless Method [1], iterative methods [2], minimization and regularization techniques [3]. The above numerical methods are accurate for solving regular integral equations and also provide satisfactory numerical results. Unfortunately, they lose their effectiveness in the case of integro-differential equations with singular kernel.

Singular integro-differential equations are difficult to solve analytically, despite this complexity, this kind of equations has a great importance and also still attracting the interest of the scientific community. Indeed, many works have been considered in the previous equations: the uniqueness of the solution [13], integral equations with logarithmic and Cauchy kernel [4], the first order singular integro-differential equations with logarithmic kernels [10] and the Cauchy kernel [11]. Numerical computation is an essential phase that uses multiple numerical methods to approximate the solution. One of the best adopted method for solving singular integro-differential equations is the projection method by using Airfoil polynomials, its advantage is to reduce the dimension of the equation and instead of solving the equation, we solve their equivalent system of linear algebraic equations that can be uniquely solved to obtain the coefficients. The authors presented in [10] the projection method using Airfoil polynomials for solving the first order integro-differential equation with logarithmic kernel and the Cauchy kernel in [11], that provides a higher accuracy than the Galerkin method. We follow those works and extend the techniques to the second order integro-differential equation with logarithmic kernel.

The following second order integro-differential equation is considered with a logarithmic kernel

$$x''(s) = y(s) - \int_{-1}^{1} \ln|t-s| x(t)dt, \qquad -1 \leq s \leq 1; \tag{1}$$

$$x'(-1) = x(1) = 0,$$

where $x(s)$ is an unknown function to be determine and $y(s)$ is the source term.

The airfoil polynomial t_n of the first kind is giving as follows:

$$t_n(s) = \frac{\cos[(n+\frac{1}{2})\arccos(s)]}{\cos(\frac{1}{2}\arccos(s))}.$$

The airfoil polynomial u_n of the second kind is giving as follows:

$$u_n(s) = \frac{\sin[(n+\frac{1}{2})\arccos(s)]}{\sin(\frac{1}{2}\arccos(s))}.$$

The paper is organized as follows. In Section 2, the discretization of the integro-differential equation by the airfoil polynomials is presented. In Section 3, the estimation of the errors were obtained on the classical Sobolev spaces. Section 4 presents some numerical examples to show the efficiency of our method. Section 5 concludes the paper.

2 The Discretization of the Integro-differential Equation

In this section, Eq(1) is transformed by Airfoil polynomials of the first kind, and here the following formulas [6] are considered as:

$$(1-s)t_i'(s) = (i+\frac{1}{2})u_i(s) - \frac{1}{2}t_i(s),$$
$$(1-s^2)t_i''(s) + (1-2s)t_i'(s) + i(i-1)u_i(s) = 0,$$

which implies that

$$t_i''(s) = -\frac{(1-2s)}{(1-s^2)}\left[\frac{(i+\frac{1}{2})}{1+s}u_i(s) - \frac{1}{2(1+s)}t_i(s)\right] - \frac{i(i-1)}{(1-s^2)}u_i(s), \tag{2}$$

and

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}t_i(t)ln|t-s|dt = \begin{cases} \frac{u_{i-1}-u_i(s)}{2(1-s^2)} + \frac{u_i+u_{i+1}(s)}{2(i+1)} & (i \neq 0), \\ -ln2 - s & (i = 0). \end{cases} \tag{3}$$

Define the approximate solution of Eq(1) as:

$$x_n(s) = w(s)\sum_{i=0}^{n}\alpha_i t_i(s),$$

where

$$w(s) = \sqrt{\frac{1+s}{1-s}}.$$

Then, Eq(1) takes the form:

$$x_n''(s) + \int_{-1}^{1} ln|t-s|x_n(t)dt = y(s). \tag{4}$$

Let $\Gamma = \{s_j = -\frac{cos(2j+1)}{2n+3},\ j = 0,\cdots,n\}$ be a set of $n+1$ collocation points which are the zeros of u_{n+1}. By substituting (2) and (3) in (4) one obtain the following linear system:

$$Av = b, \tag{5}$$

where $(a_{i,j})_{0\le i,j\le n}$ are the elements of A which are given as:

$$a_{0,j} = w''(s_j)t_0(s_j) + w'(s_j)\left[\frac{u_0(s_j)}{(1+s_j)} - \frac{t_0(s_j)}{(1+s_j)}\right] - w(s_j)\frac{(1-2s_j)}{(1+s_j)}\left[\frac{u_0(s_j)-t_0(s_j)}{2}\right] + \pi(ln2 - s_j),$$

$$a_{i,j} = w''(s_j)t_i(s_j) + 2w'(s_j)\left[\frac{(i+\frac{1}{2})u_i(s_j)}{(1+s_j)} - \frac{t_i(s_j)}{2(1+s_j)}\right] - w(s_j)\frac{(1-2s_j)}{(1+s_j)}\left[(i+\frac{1}{2})u_i(s_j) - \frac{1}{2}t_i(s_j)\right]$$
$$- w(s_j)\frac{i(i-1)}{(1-s_j^2)}u_i(s_j) + \pi\left[\frac{u_{i-1}(s_j)-u_i(s_j)}{2(1-s_j^2)} + \frac{u_i(s_j)+u_{i+1}(s_j)}{2(i+1)}\right],$$

and the vectors v and b are giving as:

$$v = (\alpha_0, \alpha_1, \cdots, \alpha_n),$$
$$b = (y(s_0), y(s_1), \cdots, y(s_n)),$$

with

$$w'(s_j) = \frac{(\frac{1}{1-s_j})^{\frac{3}{2}}}{\sqrt{s_j+1}},$$
$$w''(s_j) = \frac{2s_j+1}{(\frac{-s_j-1}{s_j-1})^{\frac{3}{2}}(s_j-1)^4}.$$

As a consequence, the coefficients α_i can be obtained by solving the linear system (5).

3 Convergence Analysis

In this section, we search for the condition in which, the norm $\|x_n - x\|_\infty$ is vanish. Here, the previous study will be recalled [11].

Let us define the space:

$$\mathcal{X} = \{x \in L^2[-1,1],\ x'' \in L^2[-1,1],\ x'(-1) = x(1) = 0\}.$$

The integral operator D' is considered and defined as follows:

$$(D'\varphi)(s) = \int_s^1 \int_{-1}^{\tau} \varphi(\tau) d\tau dt, \quad \forall \varphi \in \mathcal{X}.$$

Note that, the operator D' is compact from $\mathcal{X}$ into $\mathcal{X}$. For $x \in \mathcal{X}$, the setting is introduced as follows:

$$Dx(s) = x''(s),$$

which verify:

$$(D'D)x = -x.$$

Recall that, the integral operator T with logarithmic kernel defined as:

$$T\phi(s) = \int_{-1}^{1} ln|t-s|\phi(t)dt,$$

is bounded from $\mathcal{X}$ into itself. Thus, operator equation of Eq(1) is

$$Dx + Tx = y. \tag{6}$$

The approximate equation of Eq(6) is giving as:

$$Dx_n + Tx_n = y. \tag{7}$$

Let us define the projection operator P_n from $\mathcal{X}$ into $\mathcal{X}$, which is giving by

$$P_n\phi(\tau) = \sum_{i=0}^{n} \phi(\tau_i)e_i(\tau),$$

such that, $\Omega = \{e_0, e_1, \cdots, e_n\}$ is a set of hat functions in $\mathcal{X}$, with the orthogonality property:

$$e_i(\tau_\ell) = \delta_{i\ell} = \begin{cases} 1 & \text{if } i = \ell, \\ 0 & \text{if } i \neq \ell, \end{cases}$$

where $\delta_{i\ell}$ is the Kronecker delta.

By applying the projection P_n on T. Therefore, Eq(7) becomes

$$-x_n + D'P_nTx_n = D'y. \tag{8}$$

Let $\rho > 1$ and $H^\rho([-1, 1], \mathbb{R})$ be the Sobolev space and $\|\cdot\|_\rho$ its norm. (cf. [7])

$$(I + D'T)\left(H^\rho([-1, 1], \mathbb{R})\right) = H^\rho([-1, 1], \mathbb{R}).$$

It is known from [7] that, $\exists\, c > 0$, for all $x \in H^\rho([-1, 1], \mathbb{R})$, then one can obtained that

$$\|(I - \pi_n)x\| \leq cn^{-\rho}\|x\|_\rho.$$

Theorem 3.1. *Assume that $y \in H^\rho([-1, 1], \mathbb{R})$. Hence, the following estimate holds*

$$\|x_n - x\| \leq C_1 n^{2-\rho}\|Tx\|_{\rho-2}.$$

Proof. From (6) and (8) we get

$$x_n - x = [D'P_nTx_n - D'y] - [D'Tx - D'y][D'P_nT(x_n - x) + (D'P_nT - D'T)x],$$

then,

$$(I - D'P_nT)(x_n - x) = (D'P_nT - D'T)x.$$

Thus,

$$\begin{aligned} x_n - x &= (I - D'P_nT)^{-1}(D'P_nT - D'T)x \\ &= (I - D'P_nT)^{-1}D'(P_n - I)Tx. \end{aligned}$$

$$\begin{aligned} \|x_n - x\| &\leq \|(I - D'P_nT)^{-1}\|\ \|D'\|\ \|(P_n - I)Tx\| \\ &\leq C_1 n^{2-\rho}\ \|Tx\|_{\rho-2}, \end{aligned}$$

where $C_1 = c\,\|(I - D'P_nT)^{-1}\|\ \|D'\|$.

Hence, our desire results are obtained. □

4 Numerical Test

In order to test the numerical solution, the projection method by Airfoil polynomials provides an efficient method for the approximate solution of the singular integro-differential equations. Here, some numerical examples is presented to show the feasibility of the theoretical results obtained in the above section.

For the numerical calculations, the Gaussian quadrature rule is used to solve the linear system $Ax = b$. The errors of the collocation method are given for several values of n.

As to be expected the present method shows a satisfactory accuracy for solving the second order integro-differential equation with a logarithmic kernel as well as a high efficiency for the analytical case and also for the computed approximation.

Example 1

Starting with an exact solution to the integro-differential equation (1) as:

$$x(s) = -s^2 - 2s + 3.$$

The source term is obtained as:

$$y(s) = -2 - \pi(-s^2 - 2s + 3)(s + \ln 2)\sqrt{\frac{1-s}{1+s}}.$$

Table 1: Error analysis of $\|x_n - x\|_\infty$ for Example 1.

n	$\|x_n - x\|_\infty$
3	5.2×10^{-2}
30	1.00×10^{-2}
40	9.41×10^{-3}
50	3.90×10^{-3}
60	1.25×10^{-3}
110	1.15×10^{-3}
130	6.90×10^{-4}
150	3.75×10^{-4}
200	1.31×10^{-4}

Table 1 shows the error analysis of the method for different n.

Example 2

In the second example the exact solution to the integro-differential equation (1) as:

$$x(s) = -2s^3 + 6s - 4.$$

The source term is obtained as:

$$y(s) = -12s - \pi(-2s^3 + 6s - 4)(s + \ln 2)\sqrt{\frac{1-s}{1+s}}.$$

Table 2: Error analysis of $\|x_n - x\|_\infty$ for Example 2.

n	$\|x_n - x\|_\infty$
5	2.59×10^{-1}
20	2.90×10^{-2}
25	8.85×10^{-3}
35	3.40×10^{-4}
50	5.67×10^{-5}
90	2.20×10^{-5}
135	1.13×10^{-5}
177	5.10×10^{-6}
185	4.12×10^{-6}
195	3.17×10^{-7}

Table 2 shows the error analysis of the method for different n.

5 Conclusion

In this paper, a projection method based on Airfoil polynomials is presented for solving second order integro-differential equation of second kind with logarithmic kernel. We have demonstrated the accuracy of the proposed method through the convergence analysis and the numerical tests. We believe that the proposed method could be applied to solve other classes of singular integro-differential equations.

Acknowledgement The authors are very grateful to the editors and reviewers for carefully reading the paper and for their comments and suggestions which have improved the paper. The authors have been partially supported by The General Directorate of Scientific Research and Technological Development (DGRSDT).

Conflicts of Interest The authors declare no conflict of interest.

References

[1] P. Assari & M. Dehghan (2017). A meshless method for the numerical solution of nonlinear weakly singular integral equations using radial basis functions. *The European Physical journal Plus*, *132*, Article number: 199 (2017). https://doi.org/10.1140/epjp/i2017-11467-y.

[2] S. S. Behzadi, S. Abbasbandy & T. Allahviranloo (2013). A study on singular integro-differential equation of Abel's type by iterative methods. *Journal of Applied Mathematics & Informatics*, *31*(3-4), 499–511.

[3] A. Chakrabarti & S. R. Maman (2003). Solution of a logarithmic singular integral equation. *Applied Mathematics Letters*, *16*(3), 369–373.

[4] A. Chakrabarti & T. Sahoo (1996). Solution of singular integral equations with logarithmic and Cauchy kernels. *Proceedings of the Indian Academy of Sciences - Mathematical Sciences*, *106*, Article number: 261 (1996). https://doi.org/10.1007/BF02867434.

[5] H. Chebbah, A. Mennouni & N. Ramdani (2018). Numerical solution of generalized logarithmic integral equations of the second kind by projections. *Malaysian journal of Mathematical Sciences*, *12*(3), 349–367.

[6] R. Desmarais & S. Bland (1995). *Tables of properties of airfoil polynomials*. United States, US.

[7] W. Han & K. E. Atkinson (2009). *Theoretical numerical analysis: A functional analysis framework*. Springer, Berlin, Heidelberg.

[8] R. Kulkarni & N. Gnaneshwar (2003). Iterated discrete polynomially based Galerkin methods. *Applied Mathematics and Computation*, *146*(1), 153–165.

[9] K. H. Maleknejad & Y. Rostami (2019). B-spline method for solving Fredholm integral equations of the first kind. *International Journal of Industrial Mathematics*, *11*(1), 63–70.

[10] A. Mennouni (2012). Airfoil polynomials for solving integro-differential equations with logarithmic kernel. *Applied Mathematics and Computation*, *218*(24), 11947–11951.

[11] A. Mennouni & N. Ramdani (2018). Collocation method to solve second order Cauchy integro-differential equations. In *International Conference on Differential & Difference Equations and Applications*, pp. 307–312. Springer, Cham.

[12] B. L. P & N. Gnaneshwar (2013). Legendre Galerkin method for weakly singular Fredholm integral equations and the corresponding eigenvalue problem. *Journal of Applied Mathematics and Computing*, *43*, 175–197.

[13] T. Yuldashev & S. Zarifzoda (2020). Mellin transform and integro-differential equations with logarithmic singularity in the kernel. *Lobachevskii Journal of Mathematics*, *41*, 1910–1917.